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Transformation of certain generalized Kampé de Fériet functions

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Abstract. New transformations of generalized Kampé de Fériet functions of several variables which occur in physical and quantum chemical contexts are given. Unlike many of the known results for these functions, the variables are not fixed.

1. A multidimensional transformation

A convenient generalization of the Kampé de Fériet function was introduced by Karlsson (1973) and discussed at some length by Exton (1976, p 108). This function is denoted by

$$F^{A:B}_{C:D}\left[\begin{matrix} a_1, a_2, \ldots, a_A : b_{1,1}, b_{1,2}, \ldots, b_{1,B}; b_{2,1}, b_{2,2}, \ldots, b_{2,B}; \ldots; b_{n,1}, b_{n,2}, \ldots, b_{n,B}; \\ c_1, c_2, \ldots, c_C : d_{1,1}, d_{1,2}, \ldots, d_{1,D}; d_{2,1}, d_{2,2}, \ldots, d_{2,D}; \ldots; d_{n,1}, d_{n,2}, \ldots, d_{n,D}; \end{matrix}\right.$$
$$\left. x_1, \ldots, x_n \right] = F^{A:B}_{C:D}\left[\begin{matrix} (a) : (b_1); \ldots; (b_n); \\ (c) : (d_1); \ldots; (d_n); \end{matrix} x_1, \ldots, x_n\right]$$

with series representation

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{\left\{\prod_{j=1}^{A}(a_j, m_1 + \cdots + m_n)\right\}\left\{\prod_{j=1}^{B}(b_{1,j}, m_1)(b_{2,j}, m_2)\cdots(b_{n,j}, m_n)\right\}}{\left\{\prod_{j=1}^{C}(c_j, m_1 + \cdots + m_n)\right\}\left\{\prod_{j=1}^{D}(d_{1,j}, m_1)(d_{2,j}, m_2)\cdots(d_{n,j}, m_n)\right\}}$$
$$\times \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} = \sum \frac{((a), \Sigma m)((b_1), m_1)\cdots((b_n), m_n)x_1^{m_1}\cdots x_n^{m_n}}{((c), \Sigma m)((d_1), m_1)\cdots((d_n), m_n)m_1!\cdots m_n!} \tag{1.1}$$

in which the indices of summation run over all of the non-negative integers. The symbol (a) is a convenient contraction for the sequence of parameters $a_1, a_2, \ldots, a_A$ and the Pochhammer symbol (a, n), as usual, is given by

$$(a, n) = \Gamma(a+n)/\Gamma(a) = a(a+1)\ldots(a+n-1) \qquad (a, 0) = 1\,. \tag{1.2}$$

Also, $\Sigma m = m_1 + m_2 + \cdots + m_n$, etc, and any values of parameters for which any results do not make sense are tacitly excluded.

This study of certain transformations of generalized Kampé de Fériet functions is motivated in part by the fact that they are a more usual representation of certain n-fold hypergeometric functions studied by Niukkanen (1983, 1984), and with applications in connection with multi-centre matrix elements arising in the use of variational methods to molecular systems, for example. While a number of transformations of generalized Kampé de Fériet functions are known for fixed values of the arguments, the number of such relations in which

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the arguments are not fixed is very small in number and is largely limited to the Euler transformations of the Lauricella functions of the first and second kinds. For example, we have

$$F_D^{(n)}[a, b_1, b_2, \ldots, b_n; c; x_1, x_2, \ldots, x_n] = F_{1:0}^{1:1}\left[\begin{matrix} a : b_1, b_2, \ldots, b_n; \\ c : -; -; \cdots; -; \end{matrix} x_1, x_2, \ldots, x_n\right]$$
$$= (1 - x_1)^{-b}1(1 - x_2)^{-b}2 \ldots (1 - x_n)^{-b}n$$
$$\times F_{1:0}^{1:1}\left[\begin{matrix} c - a : b_1, b_2, \ldots, b_n; \\ c \quad : -; -; \cdots; -; \end{matrix} x_1/(x_1 - 1), x_2/(x_2 - 1), \ldots, x_n/(x_n - 1)\right] \tag{1.3}$$

as given by Appell and Kampé de Fériet (1926, part 1, chapter 7).

New transformations of a similar type may be deduced from a more general result involving multiple series of generalized Kampé de Fériet functions. This is deduced by the elementary manipulation of series as follows.

Consider the expression

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{\left\{\prod_{j=1}^{A}(a_j, m_1 + \cdots + m_n)\right\}\left\{\prod_{j=1}^{D}(d_j, m_1 + r_1 + \cdots + m_n + r_n)\right\}}{\left\{\prod_{j=1}^{H}(h_j, m_1 + \cdots + m_n)\right\}\left\{\prod_{j=1}^{G}(g_j, m_1 + r_1 + \cdots + m_n + r_n)\right\}}$$
$$\times \frac{\left\{\prod_{j=1}^{C}(c_{1,j}, m_1)(c_{2,j}, m_2) \cdots (c_{n,j}, m_n)\right\}}{\left\{\prod_{j=1}^{P}(p_{1,j}, m_1)(p_{2,j}, m_2) \cdots (p_{n,j}, m_n)\right\}}$$
$$\times \frac{\left\{\prod_{j=1}^{L}(l_{1,j}, m_1 + r_1)(l_{2,j}, m_2 + r_2) \cdots (l_{n,j}, m_n + r_n)\right\}}{\left\{\prod_{j=1}^{B}(b_{1,j}, m_1 + r_1)(b_{2,j}, m_2 + r_2) \cdots (b_{n,j}, m_n + r_n)\right\}}$$
$$\times \frac{\left\{\prod_{j=1}^{Z}(z_{1,j}, 2m_1 + r_1)(z_{2,j}, 2m_2 + r_2) \cdots (z_{n,j}, 2m_n + r_n)\right\}}{\left\{\prod_{j=1}^{K}(k_{1,j}, 2m_1 + r_1)(k_{2,j}, 2m_2 + r_2) \cdots (k_{n,j}, 2m_n + r_n)\right\}}$$
$$\times \frac{y_1^{m_1} x_1^{m_1+r_1} y_2^{m_2} x_2^{m_2+r_2} \ldots y_n^{m_n} x_n^{m_n+r_n}}{m_1! r_1! m_2! r_2! \ldots m_n! r_n!}$$
$$= \sum \frac{((a), \Sigma m)((d), \Sigma m + \Sigma r)((c_1), m_1) \cdots ((c_n), m_n)((l_1), m_1 + r_1) \cdots ((l_n), m_n + r_n)}{((h), \Sigma m)((g), \Sigma m + \Sigma r)((p_1), m_1) \cdots ((p_n), m_n)((b_1), m_1 + r_1) \cdots ((b_n), m_n + r_n)}$$
$$\times \frac{((z_1), 2m_1 + r_1) \cdots ((z_n), 2m_n + r_n) y_1^{m_1} x_1^{m_1+r_1} \cdots y_n^{m_n} x_n^{m_n+r_n}}{((k_1), 2m_1 + r_1) \cdots ((k_n), 2m_n + r_n) m_1! r_1! \ldots m_n! r_n!} \tag{1.4}$$

in the contracted notation.

If r_i is replaced by $N_i - m_i$, $1 \leqslant i \leqslant n$, then after a little re-arrangement and using the contracted notation, we have the formal result

$$\sum \frac{((a), \Sigma m)((d), \Sigma m)((c_1), m_1)((l_1), m_1)((z_1), 2m_1)}{((h), \Sigma m)((g), \Sigma m)((p_1), m_1)((b_1), m_1)((k_1), 2m_1)} \cdots$$
$$\times \frac{((c_n), m_n)((l_n), m_n)((z_n), 2m_n)(x_1 y_1)^{m_1} \cdots (x_n y_n)^{m_n}}{((p_n), m_n)((b_n), m_n)((k_n), 2m_n) m_1! \ldots m_n!}$$
$$\times F_{G:B+K}^{D:L+Z}\left[\begin{matrix} (d) + \Sigma m : (l_1) + m_1, (z_1) + 2m_1; \ldots; (l_n) + m_n, (z_n) + 2m_n; \\ (g) + \Sigma m : (b_1) + m_1, (k_1) + 2m_1; \ldots; (b_n) + m_n, (k_n) + 2m_n; \end{matrix} \right.$$
$$\left. x_1, \ldots, x_n \right]$$
$$= \sum \frac{((d), \Sigma m)((l_1), m_1)((z_1), m_1) \cdots ((l_n), m_n)((z_n), m_n) x_1^{m_1} \cdots x_n^{m_n}}{((g), \Sigma m)((b_1), m_1)((k_1), m_1) \cdots ((b_n), m_n)((k_n), m_n) m_1! \ldots m_n!}$$

$$\times F^{A:C+Z+1}_{H:P+K}\left[\begin{matrix}(a):(c_1),(z_1)+m_1,-m_1;\ldots;(c_n),(z_n)+m_n,-m_n;\\(h):\quad(p_1),(k_1)+m_1\quad;\ldots;\quad(p_n),(k_n)+m_n\quad;\end{matrix}\right.$$

$$\left.-y_1,\ldots,-y_n\right]. \tag{1.5}$$

This expression is a special case of a formula obtained by Exton (1976, page 140) by a different method. The procedure now used consists of expressing the inner generalized Kampé de Fériet functions in compact form, thus obtaining the transformations sought. Any questions of convergence must be dealt with in each individual case as appropriate.

The following formulae are required in the subsequent analysis:

$$_1F_0[a;-;x]=(1-x)^{-a}\qquad\text{(the binomial theorem)} \tag{1.6}$$

$$_2F_1[2a,a+1;a;x]=(1-x)^{-2a-1}(1+x) \tag{1.7}$$

$$F^{A:0}_{C:0}\left[\begin{matrix}(a):-;\\(c):-;\end{matrix}x_1,\ldots,x_n\right]={}_AF_C[(a);(c);\Sigma x] \tag{1.8}$$

$$F^{A:1}_{C:0}\left[\begin{matrix}(a):b_1;\ldots;b_n;\\(c):-;\ldots;-;\end{matrix}x,\ldots,x\right]={}_{A+1}F_C[(a),\Sigma b;(c);x] \tag{1.9}$$

$$F^{0:B}_{0:D}\left[\begin{matrix}-:(b_1):\ldots:(b_n);\\-:(d_1);\ldots;(d_n);\end{matrix}x_1,\ldots,x_n\right]={}_BF_D[(b_1);(d_1);x_1]\ldots{}_BF_D[(b_n);(d_n);x_n] \tag{1.10}$$

$$_2F_1[a,-n;c;1]=(c-a,n)/(c,n)\qquad\text{(Vandermonde's theorem)} \tag{1.11}$$

$$_3F_2[a,1+\tfrac{1}{2}a,-n;\tfrac{1}{2}a,b;1]=\frac{(2+a-b,n)(b-a-1,n)}{(b,n)(1+a-b,n)(b-a-1)} \tag{1.12}$$

$$_3F_2[a,b,-n;c,a+b-c-n+1;1]=\frac{(c-a,n)(c-b,n)}{(c,n)(c-a-b,n)}\qquad\text{(Saalschütz's theorem)} \tag{1.13}$$

$$_3F_2[a,b,c;\tfrac{1}{2}+\tfrac{1}{2}a+\tfrac{1}{2}b,2c;1]=\frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}$$

$$\text{(Watson's theorem)} \tag{1.14}$$

and

$$_4F_3\left[\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2},b+n,-n;\tfrac{1}{2}b,\tfrac{1}{2}+\tfrac{1}{2}b,1+a;1\right]=(b-a,n)/(b,n). \tag{1.15}$$

The results (1.6), (1.11)–(1.15) are to be found in Slater (1966, appendix III). For equation (1.7) see Erdélyi (1953, vol I, p 101), and for (1.8)–(1.10), see Srivastava and Karlsson (1973, p 39).

2. Transformations obtainable from the binomial theorem

In the general result (1.5), let $D=1$, suppress A, Z, H, G and K, leaving C and F unspecified. The inner Kampé de Fériet function on the left becomes

$$F^{1:0}_{0:0}\left[\begin{matrix}d+\Sigma m:-;\\-\quad:-;\end{matrix}x_1,\ldots,x_n\right] \tag{2.1}$$

which in turn reduces to

$$_1F_0[d+\Sigma m;-;\Sigma x]=(1-\Sigma x)^{d-\Sigma m}\qquad|\Sigma x|<1 \tag{2.2}$$

by making use of (1.8) and the binomial theorem. Hence, (1.5) now takes the form

$$(1-\Sigma x)^{-d} F^{1:C}_{0:P}\left[\begin{matrix} d:(c_1);\ldots;(c_n); \\ -:(p_1);\ldots;(p_n); \end{matrix} x_1y_1/(1-\Sigma x),\ldots,x_ny_n/(1-\Sigma x)\right]$$
$$= \sum \frac{(d,\Sigma m)x_1^{m_1}\cdots x_n^{m_n}}{m_1!\ldots m_n!}\, {}_{C+1}F_P\left[\begin{matrix}(c_1),-m_1; \\ (p_1) \quad ; \end{matrix} -y_1\right]\ldots$$
$$\times {}_{C+1}F_P\left[\begin{matrix}(c_n),-m_n; \\ (p_n) \quad ; \end{matrix} -y_n\right] \tag{2.3}$$

by applying (1.10) to the inner generalized Kampé de Fériet function on the right of (1.5).

If $C = P = 1$ and $y_i = -1$, each of the hypergeometric functions on the right of (2.3) can be summed by means of Vandermonde's theorem (1.11). The result obtained is not new, however, and is an Euler transformation of the Lauricella function $F_A^{(n)}$.

Now let $C = P = 2$, and consider

$$ {}_3F_2[c_1, c_2, -m; p_1, p_2; 1]. \tag{2.4}$$

Let $c_1 = 1 + \frac{1}{2}c_1$, $p_1 = \frac{1}{2}c_1$ and replace p_2 by p_1. The formula (1.12) may be used and the product of the hypergeometric functions on the right of (2.3) becomes

$$\frac{(2+c_1-p_1,m_1)(p_1-c_1-1,m_1)}{(p_1,m_1)(1+c_1-p_1,m_1)(p_1-c_1-1)}\cdots\frac{(2+c_n-p_n,m_n)(p_n-c_n-1,m_n)}{(p_n,m_n)(1+c_n-p_n,m_n)(p_n-c_n-1)}. \tag{2.5}$$

Hence,

$$(1-\Sigma x)^{-d} F^{1:2}_{0:2}\left[\begin{matrix} d: c_1, 1+\frac{1}{2}c_1;\ldots;c_n, 1+\frac{1}{2}c_n; \\ -: \quad \frac{1}{2}c_1, p_1 \quad ;\ldots; \quad \frac{1}{2}c_2, p_n \quad ; \end{matrix} x_1/(\Sigma x-1),\ldots,x_n/(\Sigma x-1)\right]$$
$$= (p_1-c_1-1)^{-1}\cdots(p_n-c_n-1)^{-1}$$
$$\times F^{1:2}_{0:2}\left[\begin{matrix} d: 2+c_1-p_1, p_1-c_1-1;\ldots;2+c_n-p_n, p_n-c_n-1; \\ -: \quad p_1, 1+c_1-p_1 \quad ;\ldots; \quad p_n, 1+c_n-p_n \quad ; \end{matrix}\right.$$
$$\left. x_1,\ldots,x_n \right]. \tag{2.6}$$

This expression is valid in the region common to $|\Sigma x| < 1$ and $|\Sigma x/(\Sigma x-1)| < 1$.

A rather curious result now follows if we put $p_i = 2 + c_i$, when the expression on the right of (2.6) reduces to unity and we see that

$$(1-\Sigma x)^{-d} F^{1:2}_{0:2}\left[\begin{matrix} d: c_1, 1+\frac{1}{2}c_1;\ldots;c_n, 1+\frac{1}{2}c_n; \\ -: \frac{1}{2}c_1, 2+c_1;\ldots;\frac{1}{2}c_n, 2+c_n; \end{matrix} x_1/(\Sigma x-1),\ldots,x_n/(\Sigma x-1)\right] = 1 \tag{2.7}$$

is convergent when $|\Sigma x/(\Sigma x-1)| < 1$.

The binomial theorem may again be used in a slightly different way. Put $L = 1$, suppress D, C, P, G, Z and B in (1.5) and if A and H are unspecified, we see that

$$(1-x_1)^{-1}1\ldots(1-x_n)^{-1}n F^{A:1}_{H:0}\left[\begin{matrix} (a): l_1;\ldots;l_n; \\ (h): -;\ldots;-; \end{matrix} x_1y_1/(1-x_1),\ldots,x_ny_n/(1-x_n)\right]$$
$$= \sum \frac{(l_1,m_1)\cdots(l_n,m_n)x_1^{m_1}\cdots x_n^{m_n}}{m_1!\ldots m_n!}$$
$$\times F^{A:1}_{H:0}\left[\begin{matrix} (a): -m_1;\ldots;-m_n; \\ (h): \quad - \quad ;\ldots; \quad - \quad ; \end{matrix} -y_1,\ldots,-y_n\right]. \tag{2.8}$$

Vandermonde's theorem again gives rise to a known result if $A = H = 1$. On putting $A = H = 2$, the summation formula (1.12) may be applied after suitable specialization to give the expression

$$(1-x_1)^{-1}1\cdots(1-x_n)^{-1}nF^{2:1}_{2:0}\left[\begin{matrix} a, 1+\frac{1}{2}a : l_1; \ldots; l_n; \\ \frac{1}{2}, a, h \quad : -; \ldots; -; \end{matrix} x_1/(x_1-1), \ldots, x_n/(x_n-1)\right]$$
$$= (h-a-1)^{-1}F^{2:1}_{2:0}\left[\begin{matrix} 2+a-h, h-a-1 : l_1; \ldots; l_n; \\ h, 1+a-h \quad : -; \ldots; -; \end{matrix} x_1, \ldots, x_n\right]$$
$$\text{for } |x_i/(x_i-1)| < 1 \text{ and } |x_i| < 1, 1 \leqslant i \leqslant n\,. \tag{2.9}$$

If $h = 2 + a$, we have the special case

$$(1-x_1)^{-1}1\ldots(1-x_n)^{-1}nF^{2:1}_{2:0}\left[\begin{matrix} a, 1+\frac{1}{2}a : l_1; \ldots; l_n; \\ \frac{1}{2}a, 2+a : -; \ldots; -; \end{matrix} x_1/(x_1-1), \ldots, x_n/(x_n-1)\right]$$
$$= 1 \qquad \text{for } |x_i/(x_i-1)| < 1\,. \tag{2.10}$$

3. Transformations deduced from (1.7)

If we let $x_1 = \cdots = x_n = x$, $D = G = Z = 1$, C and P remaining unspecified while the other groups of parameters are suppressed, the inner series on the left of (1.5) takes the form

$$F^{1:1}_{1:0}\left[\begin{matrix} d+\Sigma m : z_1+2m_1; \ldots; z_n+2m_n; \\ g+\Sigma m : \quad - \quad ; \ldots; \quad - \quad ; \end{matrix} x, \ldots, x\right] = {}_2F_1[d+\Sigma m, \Sigma z + 2\Sigma m; g+\Sigma m; x] \tag{3.1}$$

by (1.9). If, further, $d = 1 + \frac{1}{2}\Sigma z$ and $g = \frac{1}{2}\Sigma z$, (1.7) may be applied to (3.1) and its right-hand member becomes

$$(1-x)^{-1-\Sigma z-2\Sigma m}(1+x)\,. \tag{3.2}$$

On substituting this result into (1.5), we see that

$$(1-x)^{-1-\Sigma z}(1+x) \times F^{1:C+2}_{1:P}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z : (c_1), \frac{1}{2}z_1, \frac{1}{2}+\frac{1}{2}z_1; \ldots; (c_n), \frac{1}{2}z_n\frac{1}{2}+\frac{1}{2}z_n; \\ \frac{1}{2}\Sigma z \quad : \quad (p_1) \quad ; \ldots; \quad (p_n) \quad ; \end{matrix}\right.$$
$$\left. 4xy_1(1-x)^{-2}, \ldots, 4xy_n(1-x)^{-2}\right]$$
$$= \sum \frac{(1+\frac{1}{2}\Sigma z, \Sigma m)(z_1, m_1)\cdots(z_n, m_n)}{(\frac{1}{2}\Sigma z, \Sigma m)m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}$$
$$\times {}_{C+2}F_P[(c_1), z_1+m_1, -m_1; (p_1); -y_1]\ldots$$
$${}_{C+2}F_P[(c_n), z_n+m_m, -m_n; (p_n); -y_n]\,. \tag{3.3}$$

Let $y_i = -1$ and apply Vandermonde's theorem, such that, if $C = 0$ and $P = 1$

$${}_2F_1[z_i+m_i, -m_i; p_i; 1] = (p_i - z_i - m_i, m_i)/(p_i, m_i)$$
$$= (1+z_i-p_i, m_i)(-1)^m i/(p_i, m_i)\,. \tag{3.4}$$

We may then write down the formula

$$(1-x)^{-1-\Sigma z}(1+x)F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z : \frac{1}{2}z_1, \frac{1}{2}+\frac{1}{2}z_1; \ldots; \frac{1}{2}z_n, \frac{1}{2}+\frac{1}{2}z_n; \\ \frac{1}{2}\Sigma z \quad : \quad p_1 \quad ; \ldots; \quad p_n \quad ; \end{matrix}\right.$$
$$\left. -4x(1-x)^{-2}, \ldots, -4x(1-x)^2\right]$$

$$= F^{1:2}_{1:1}\left[\begin{matrix} 1+\Sigma z: z_1, 1+z_1-p_1; \ldots; z_n, 1+z_n-p_n; \\ \frac{1}{2}\Sigma z \quad : \quad p_1 \quad ; \ldots ; \quad p_n \quad ; \end{matrix} -x, \ldots, -x\right] \tag{3.5}$$

such that $|x| < 1$ and $|x(1-x)^{-2}| < \frac{1}{4}$.

Two special cases of this expression are worth mentioning, namely, if $p_i = 1 + z_i$,

$$F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: \frac{1}{2}z_1, \frac{1}{2}+\frac{1}{2}z_1; \ldots; \frac{1}{2}z_n, \frac{1}{2}+\frac{1}{2}z_n; \\ \frac{1}{2}\Sigma z \quad : \quad 1+z_1 \quad ; \ldots ; \quad 1+z_n \quad ; \end{matrix} -4x(1-x)^{-2}, \ldots, -4x(1-x)^{-2}\right]$$
$$= (1-x)^{1+\Sigma z}(1+x)^{-1} \tag{3.6}$$

and if $p_i = \frac{1}{2}z_i$,

$$F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: z_1, 1+\frac{1}{2}z_1; \ldots; z_n, 1+\frac{1}{2}z_n; \\ \frac{1}{2}\Sigma z \quad : \quad \frac{1}{2}z_1 \quad ; \ldots ; \quad \frac{1}{2}z_n \quad ; \end{matrix} -x, \ldots, -x\right]$$
$$= (1-x)^{-1-\Sigma z}(1+x)\,{}_2F_1\left[1+\tfrac{1}{2}\Sigma z, \tfrac{1}{2}n+\tfrac{1}{2}\Sigma z; \tfrac{1}{2}\Sigma z; -4x(1-x)^{-2}\right]. \tag{3.7}$$

Saalschütz's theorem (1.13), Watson's theorem (1.14) and the formula (1.15) may now be used in a similar fashion to simplify the right-hand member of (3.3). We now obtain, respectively, the expressions

$$F^{1:3}_{1:2}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: \quad c_1, \frac{1}{2}z_1, \frac{1}{2}+\frac{1}{2}z_1 \quad ; \ldots ; \quad c_n, \frac{1}{2}z_n, \frac{1}{2}+\frac{1}{2}z_n \quad ; \\ \frac{1}{2}\Sigma z \quad : p_1, c_1+z_1+1-p_1; \ldots; p_n, c_n+z_n+1-p_n; \end{matrix}\right.$$
$$\left. -4x(1-x)^{-2}, \ldots, -4x(1-x)^{-2}\right]$$
$$= (1-x)^{1+\Sigma z}(1+x)^{-1}$$
$$\times F^{1:3}_{1:2}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: z_1, p_1-c_1, 1+z_1-p_1; \ldots; z_n, p_n-c_n, 1+z_n-p_n; \\ \frac{1}{2}\Sigma z \quad : \quad p_1, 1+z_1+c_1-p_1 \quad ; \ldots ; \quad p_n, 1+z_n+c_n-p_n \quad ; \end{matrix}\right.$$
$$\left. x, \ldots, x\right] \tag{3.8}$$

$$F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: c_1, \frac{1}{2}z_1; \ldots; c_n, \frac{1}{2}z_n; \\ \frac{1}{2}\Sigma z \quad : \quad 2c_1 \quad ; \ldots ; \quad 2c_n \quad ; \end{matrix} -4x(1-x)^{-2}, \ldots, -4x(1-x)^{-2}\right]$$
$$= (1-x)^{1+\Sigma z}(1+x)F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{4}\Sigma z: \frac{1}{2}z_1, \frac{1}{2}+\frac{1}{2}z_1-c_1; \ldots; \frac{1}{2}z_n, \frac{1}{2}+\frac{1}{2}z_n; \\ \frac{1}{4}\Sigma z \quad : \quad \frac{1}{2}+c_1 \quad ; \ldots ; \quad \frac{1}{2}+c_n \quad ; \end{matrix}\right.$$
$$\left. x, \ldots, x\right] \tag{3.9}$$

and

$$F^{1:2}_{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: \frac{1}{2}c_1, \frac{1}{2}+\frac{1}{2}c_1; \ldots; \frac{1}{2}c_n, \frac{1}{2}+\frac{1}{2}c_n; \\ \frac{1}{2}\Sigma z \quad : \quad 1+c_1 \quad ; \ldots ; \quad 1+c_n \quad ; \end{matrix} -4x(1-x)^{-2}, \ldots, -4x(1-x)^{-2}\right]$$
$$= (1-x)^{1+\Sigma z}(1+x)^{-1}F^{1:1}\left[\begin{matrix} 1+\frac{1}{2}\Sigma z: z_1-c_1; \ldots; z_n-c_n; \\ \frac{1}{2}\Sigma z \quad : \quad - \quad ; \ldots ; \quad - \quad ; \end{matrix} x, \ldots, x\right]$$
$$= (1-x)^{1+\Sigma z}(1+x)^{-1}\,{}_2F_1\left[1+\tfrac{1}{2}\Sigma z, \Sigma z - \Sigma c; \tfrac{1}{2}\Sigma z; x\right] \tag{3.10}$$

under the same convergence conditions as those which apply to (3.5). It must be stressed that the list of transformations given in this study is by no means exhaustive.

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